

# THE COVERING NUMBER OF THE DIFFERENCE SETS IN PARTITIONS OF $G$ -SPACES AND GROUPS

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**ABSTRACT.** We prove that for every finite partition  $G = A_1 \cup \dots \cup A_n$  of a group  $G$  either  $\text{cov}(A_i A_i^{-1}) \leq n$  for all cells  $A_i$  or else  $\text{cov}(A_i A_i^{-1} A_i) < n$  for some cell  $A_i$  of the partition. Here  $\text{cov}(A) = \min\{|F| : F \subset G, G = FA\}$  is the covering number of  $A$  in  $G$ . A similar result is proved also of partitions of  $G$ -spaces. This gives two partial answers to a problem of Protasov posed in 1995.

This paper was motivated by the following problem posed by I.V.Protasov in Kourouka Notebook [7].

**Problem 1** (Protasov, 1995). *Is it true that for any partition  $G = A_1 \cup \dots \cup A_n$  of a group  $G$  some cell  $A_i$  of the partition has  $\text{cov}(A_i A_i^{-1}) \leq n$ ?*

Here for a non-empty subset  $A \subset G$  by

$$\text{cov}(A) = \min\{|F| : F \subset G, G = FA\}$$

we denote the *covering number* of  $A$ .

In fact, Protasov's Problem can be posed in a more general context of ideal  $G$ -spaces. Let us recall that a  $G$ -space is a set  $X$  endowed with an action  $G \times X \rightarrow X$ ,  $(g, x) \mapsto gx$ , of a group  $G$ . An *ideal  $G$ -space* is a pair  $(X, \mathcal{I})$  consisting of a  $G$ -space  $X$  and a  $G$ -invariant Boolean ideal  $\mathcal{I} \subset \mathcal{B}(X)$  in the Boolean algebra  $\mathcal{B}(X)$  of all subsets of  $X$ . A *Boolean ideal* on  $X$  is a proper subfamily  $\mathcal{I} \subsetneq \mathcal{B}(X)$  such that for any  $A, B \in \mathcal{I}$  any subset  $C \subset A \cup B$  belongs to  $\mathcal{I}$ . A Boolean ideal  $\mathcal{I}$  is  $G$ -invariant if  $\{gA : g \in G, A \in \mathcal{I}\} \subset \mathcal{I}$ . A Boolean ideal  $\mathcal{I} \subset \mathcal{B}(G)$  on a group  $G$  will be called *invariant* if  $\{xAy : x, y \in G, A \in \mathcal{I}\} \subset \mathcal{I}$ . By  $[X]^{<\omega}$  and  $[X]^{\leq\omega}$  we denote the families of all finite and countable subsets of a set  $X$ , respectively. The family  $[X]^{<\omega}$  (resp.  $[X]^{\leq\omega}$ ) is a Boolean ideal on  $X$  if  $X$  is infinite (resp. uncountable).

For a subset  $A \subset X$  of an ideal  $G$ -space  $(X, \mathcal{I})$  by

$$\Delta(A) = \{g \in G : gA \cap A \neq \emptyset\} \text{ and } \Delta_{\mathcal{I}}(A) = \{g \in G : gA \cap A \notin \mathcal{I}\}$$

we denote the *difference set* and  $\mathcal{I}$ -*difference set* of  $A$ , respectively.

Given a Boolean ideal  $\mathcal{J}$  on a group  $G$  and two subsets  $A, B \subset G$  we shall write  $A =_{\mathcal{J}} B$  if the symmetric difference  $A \Delta B = (A \setminus B) \cup (B \setminus A)$  belongs to the ideal  $\mathcal{J}$ . For a non-empty subset  $A \subset G$  put

$$\text{cov}_{\mathcal{J}}(A) = \min\{|F| : F \subset G, FA =_{\mathcal{J}} G\}$$

be the  $\mathcal{J}$ -*covering number* of  $A$ . For the empty subset we put  $\text{cov}_{\mathcal{J}}(\emptyset) = \infty$  and assume that  $\infty$  is larger than any cardinal number.

Observe that for the left action of the group  $G$  on itself we get  $\Delta(A) = AA^{-1}$  for every subset  $A \subset G$ . That is why Problem 1 is a partial case of the following general problem.

**Problem 2.** *Is it true that for any partition  $X = A_1 \cup \dots \cup A_n$  of an ideal  $G$ -space  $X$  some cell  $A_i$  of the partition has  $\text{cov}(\Delta_{\mathcal{I}}(A_i)) \leq n$ ?*

This problem has an affirmative answer for  $G$ -spaces with amenable acting group  $G$ , see [3, 4.3]. The paper [3] gives a survey of available partial solutions of Protasov's Problems 1 and 2. Here we mention the following result of Banach, Ravsky and Slobodianiuk [1].

**Theorem 1.** *For any partition  $X = A_1 \cup \dots \cup A_n$  of an ideal  $G$ -space  $(X, \mathcal{I})$  some cell  $A_i$  of the partition has*

$$\text{cov}(\Delta_{\mathcal{I}}(A_i)) \leq \max_{0 < k \leq n} \sum_{p=0}^{n-k} k^p \leq n!$$

In this paper we shall give another two partial solutions to Protasov's Problems 1 and 2.

**Theorem 2.** *For any partition  $X = A_1 \cup \dots \cup A_n$  of an ideal  $G$ -space  $(X, \mathcal{I})$  either*

- $\text{cov}(\Delta_{\mathcal{I}}(A_i)) \leq n$  for all cells  $A_i$  or else

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- $\text{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) < n$  for some cell  $A_i$  and some  $G$ -invariant ideal  $\mathcal{J} \not\supseteq \Delta_{\mathcal{I}}(A_i)$  on  $G$ .

**Corollary 1.** *For any partition  $X = A_1 \cup \dots \cup A_n$  of an ideal  $G$ -space  $(X, \mathcal{I})$  either  $\text{cov}(\Delta_{\mathcal{I}}(A_i)) \leq n$  for all cells  $A_i$  or else  $\text{cov}(\Delta_{\mathcal{I}}(A_i) \cdot \Delta_{\mathcal{I}}(A_i)) < n$  for some cell  $A_i$ .*

*Proof.* By Theorem 2, either  $\text{cov}(\Delta_{\mathcal{I}}(A_i)) \leq n$  for all cells  $A_i$  or else there is a cell  $A_i$  of the partition such that  $\text{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) < n$  for some  $G$ -invariant ideal  $\mathcal{J} \subsetneq \mathcal{P}(G)$ . In the first case we are done. In the second case we can find a subset  $F \subset G$  of cardinality  $|F| < n$  such that  $F \cdot \Delta_{\mathcal{I}}(A_i) =_{\mathcal{J}} G$ . It follows from  $G \notin \mathcal{J} \supset G \setminus (F \cdot \Delta_{\mathcal{I}}(A_i))$  that  $F \cdot \Delta_{\mathcal{I}}(A_i) \notin \mathcal{J}$  and hence  $\Delta_{\mathcal{I}}(A_i) \notin \mathcal{J}$  by the  $G$ -invariance of  $\mathcal{J}$ . Then for every  $x \in G$  the shift  $x\Delta_{\mathcal{I}}(A_i)$  does not belong to the ideal  $\mathcal{J}$  and hence intersects the set  $F \cdot \Delta_{\mathcal{I}}(A_i)$ . So  $x \in F \cdot \Delta_{\mathcal{I}}(A_i) \cdot \Delta_{\mathcal{I}}(A_i)^{-1} = F \cdot \Delta_{\mathcal{I}}(A_i) \cdot \Delta_{\mathcal{I}}(A_i)$  and  $\text{cov}(\Delta_{\mathcal{I}}(A_i) \cdot \Delta_{\mathcal{I}}(A_i)) \leq |F| \leq n$ .  $\square$

For groups  $G$  (considered as  $G$ -spaces endowed with the left action of  $G$  on itself), we can prove a bit more:

**Theorem 3.** *Let  $G$  be a group and  $\mathcal{I}$  be an invariant Boolean ideal on  $G$  with  $[G]^{\leq \omega} \not\subset \mathcal{I}$ . For any partition  $G = A_1 \cup \dots \cup A_n$  of  $G$  either*

- $\text{cov}(\Delta_{\mathcal{I}}(A_i)) \leq n$  for all cells  $A_i$  or else
- $\text{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) < n$  for some cell  $A_i$  and for some  $G$ -invariant Boolean ideal  $\mathcal{J} \not\supseteq A_i^{-1}$  on  $G$ .

**Corollary 2.** *For any partition  $G = A_1 \cup \dots \cup A_n$  of a group  $G$  either  $\text{cov}(A_i A_i) \leq n$  for all cells  $A_i$  or else  $\text{cov}(A_i A_i^{-1} A_i) < n$  for some cell  $A_i$  of the partition.*

*Proof.* On the group  $G$  consider the trivial ideal  $\mathcal{I} = \{\emptyset\}$ . By Theorem 3, either  $\text{cov}(A_i A_i^{-1}) \leq n$  for all cells  $A_i$  or else  $\text{cov}_{\mathcal{J}}(A_i A_i^{-1}) < n$  for some cell  $A_i$  and some  $G$ -invariant ideal  $\mathcal{J} \not\supseteq A_i^{-1}$  on  $G$ . In the first case we are done. In the second case, choose a finite subset  $F \subset G$  of cardinality  $|F| < n$  such that the set  $F A_i A_i^{-1} =_{\mathcal{J}} G$ . Since  $A_i^{-1} \notin \mathcal{J}$ , for every  $x \in G$  the set  $x A_i^{-1}$  intersects  $F A_i A_i^{-1}$  and thus  $x \in F A_i A_i^{-1} A_i$  and  $\text{cov}(A_i A_i^{-1} A_i) \leq |F| < n$ .  $\square$

Taking into account that the ideal  $\mathcal{J}$  appearing in Theorem 3 is  $G$ -invariant but not necessarily invariant, we can ask the following question.

**Problem 3.** *Is it true that for any partition  $G = A_1 \cup \dots \cup A_n$  of a group  $G$  some cell  $A_i$  of the partition has  $\text{cov}_{\mathcal{J}}(A_i A_i^{-1}) \leq n$  for some invariant Boolean ideal  $\mathcal{J}$  on  $G$ ?*

## 1. MINIMAL MEASURES ON $G$ -SPACES

Theorems 2 and 3 will be proved with help of minimal probability measures on  $X$  and right quasi-invariant idempotent measures on  $G$ .

For a  $G$ -space  $X$  by  $P(X)$  we denote the (compact Hausdorff) space of all finitely additive probability measures on  $X$ . The action of the group  $G$  on  $X$  extends to an action of the convolution semigroup  $P(G)$  on  $P(X)$ : for two measures  $\mu \in P(G)$  and  $\nu \in P(X)$  their convolution is defined as the measure  $\mu * \nu \in P(X)$  assigning to each bounded function  $\varphi : X \rightarrow \mathbb{R}$  the real number

$$\mu * \nu(\varphi) = \int_G \int_X \varphi(g^{-1}x) d\nu(x) d\mu(g).$$

The convolution map  $* : P(G) \times P(X) \rightarrow P(X)$  is right-continuous in the sense that for any fixed measure  $\nu \in P(X)$  the right shift  $P(G) \rightarrow P(X)$ ,  $\mu \mapsto \mu * \nu$ , is continuous. This implies that the  $P(G)$ -orbit  $P(G) * \nu = \{\mu * \nu : \mu \in P(G)\}$  of  $\nu$  coincides with the closure  $\overline{\text{conv}}(G \cdot \nu)$  of the convex hull of the  $G$ -orbit  $G \cdot \nu$  of  $\nu$  in  $P(X)$ .

A measure  $\mu \in P(X)$  will be called *minimal* if for any measure  $\nu \in P(G) * \mu$  we get  $P(G) * \nu = P(G) * \mu$ . The Zorn's Lemma combined with the compactness of the orbits implies that the orbit  $P(G) * \mu$  of each measure  $\mu \in P(X)$  contains a minimal measure.

It follows from Day's Fixed Point Theorem [8, 1.14] that for a  $G$ -space  $X$  with amenable acting group  $G$  each minimal measure  $\mu$  on  $X$  is  $G$ -invariant, which implies that the set  $\overline{\text{conv}}(G \cdot \mu)$  coincides with the singleton  $\{\mu\}$ .

For an ideal  $G$ -space  $(X, \mathcal{I})$  let  $P_{\mathcal{I}}(X) = \{\mu \in P(X) : \forall A \in \mathcal{I} \mu(A) = 0\}$ .

**Lemma 1.** *For any ideal  $G$ -space  $(X, \mathcal{I})$  the set  $P_{\mathcal{I}}(X)$  contains some minimal probability measure.*

*Proof.* Let  $\mathcal{U}$  be any ultrafilter on  $X$ , which contains the filter  $\mathcal{F} = \{F \subset X : X \setminus F \in \mathcal{I}\}$ . This ultrafilter  $\mathcal{U}$  can be identified with the 2-valued measure  $\mu_{\mathcal{U}} : \mathcal{B}(X) \rightarrow \{0, 1\}$  such that  $\mu_{\mathcal{U}}^{-1}(1) = \mathcal{U}$ . It follows that  $\mu_{\mathcal{U}}(A) = 0$  for any subset  $A \in \mathcal{I}$ . In the  $P(G)$ -orbit  $P(G) * \mu_{\mathcal{U}}$  choose any minimal measure  $\mu = \nu * \mu_{\mathcal{U}}$  and observe that for every  $A \in \mathcal{I}$  the  $G$ -invariance of the ideal  $\mathcal{I}$  implies  $\mu(A) = \int_G \mu_{\mathcal{U}}(x^{-1}A) d\nu(x) = 0$ . So,  $\mu \in P_{\mathcal{I}}(X)$ .  $\square$

For a subset  $A$  of a group  $G$  put

$$\text{ls}_{12}(A) = \inf_{\mu \in P(G)} \sup_{y \in G} \mu(Ay).$$

**Lemma 2.** *If a subset  $A$  of a group  $G$  has  $\text{ls}_{12}(A) = 1$ , then  $\text{cov}(G \setminus A) \geq \omega$ .*

*Proof.* It suffices to show that  $G \neq F(G \setminus A)$  for any finite set  $F \subset G$ . Consider the uniformly distributed measure  $\mu = \frac{1}{|F|} \sum_{x \in F} \delta_{x^{-1}}$  on the set  $F^{-1}$ . Since  $\text{ls}_{12}(A) = 1$ , for the measure  $\mu$  there is a point  $y \in G$  such that  $1 - \frac{1}{|F|} < \mu(Ay) = \frac{1}{|F|} \sum_{x \in F} \delta_{x^{-1}}(Ay)$ , which implies that  $\mu(Ay) = 1$  and  $\text{supp}(\mu) = F^{-1} \subset Ay$ . Then  $F^{-1}y^{-1} \cap (G \setminus A) = \emptyset$  and  $y^{-1} \notin F(G \setminus A)$ .  $\square$

**Remark 1.** By Theorem 3.8 of [2], for every subset  $A$  of a group  $G$  we get  $\text{ls}_{12}(A) = 1 - \text{is}_{21}(G \setminus A)$  where  $\text{is}_{21}(B) = \inf_{\mu \in P_\omega(G)} \sup_{x \in G} \mu(xB)$  for  $B \subset G$  and  $P_\omega(G)$  denotes the set of finitely supported probability measures on  $G$ .

For a probability measure  $\mu \in P(X)$  on a  $G$ -space  $X$  and a subset  $A \subset X$  put

$$\bar{\mu}(A) = \sup_{x \in G} \mu(xA).$$

## 2. A DENSITY VERSION OF THEOREM 2

In this section we shall prove the following density theorem, which will be used in the proof of Theorem 2 presented in the next section.

**Theorem 4.** *Let  $(X, \mathcal{I})$  be an ideal  $G$ -space and  $\mu \in P_{\mathcal{I}}(X)$  be a minimal measure on  $X$ . If some subset  $A \subset X$  has  $\bar{\mu}(A) > 0$ , then the  $\mathcal{I}$ -difference set  $\Delta_{\mathcal{I}}(A)$  has  $\mathcal{J}$ -covering number  $\text{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A)) \leq 1/\bar{\mu}(A)$  for some  $G$ -invariant ideal  $\mathcal{J} \not\supset \Delta_{\mathcal{I}}(A)$  on  $G$ .*

*Proof.* By the compactness of  $P(G) * \mu = \overline{\text{conv}}(G \cdot \mu)$ , there is a measure  $\mu' \in P(G) * \mu \subset P_{\mathcal{I}}(X)$  such that  $\mu'(A) = \sup\{\nu(A) : \nu \in P(G) * \mu\} = \bar{\mu}(A)$ . We can replace the measure  $\mu$  by  $\mu'$  and assume that  $\mu(A) = \bar{\mu}(A)$ . Choose a positive  $\varepsilon$  such that  $\lfloor \frac{1}{\bar{\mu}(A) - \varepsilon} \rfloor = \lfloor \frac{1}{\bar{\mu}(A)} \rfloor$ , where  $\lfloor r \rfloor = \max\{n \in \mathbb{Z} : n \leq r\}$  denotes the integer part of a real number  $r$ .

Consider the set  $L = \{x \in G : \mu(xA) > \bar{\mu}(A) - \varepsilon\}$  and choose a maximal subset  $F \subset L$  such that  $\mu(xA \cap yA) = 0$  for any distinct points  $x, y \in L$ . The additivity of the measure  $\mu$  implies that  $1 \geq \sum_{x \in F} \mu(xA) > |F|(\bar{\mu}(A) - \varepsilon)$  and hence  $|F| \leq \lfloor \frac{1}{\bar{\mu}(A) - \varepsilon} \rfloor = \lfloor \frac{1}{\bar{\mu}(A)} \rfloor \leq \frac{1}{\bar{\mu}(A)}$ . By the maximality of  $F$ , for every  $x \in L$  there is  $y \in F$  such that  $\mu(xA \cap yA) > 0$ . Then  $xA \cap yA \notin \mathcal{I}$  and  $y^{-1}x \in \Delta_{\mathcal{I}}(A)$ . It follows that  $x \in y \cdot \Delta_{\mathcal{I}}(A) \subset F \cdot \Delta_{\mathcal{I}}(A)$  and  $L \subset F \cdot \Delta_{\mathcal{I}}(A)$ .

We claim that  $\text{ls}_{12}(L) = 1$ . Given any measure  $\nu \in P(G)$ , consider the measure  $\nu^{-1} \in P(G)$  defined by  $\nu^{-1}(B) = \nu(B^{-1})$  for every subset  $B \subset G$ . By the minimality of  $\mu$ , we can find a measure  $\eta \in P(G)$  such that  $\eta * \nu^{-1} * \mu = \mu$ . Then

$$\begin{aligned} \bar{\mu}(A) &= \mu(A) = \eta * \nu^{-1} * \mu(A) = \int_G \mu(x^{-1}A) d\eta * \nu^{-1}(x) \leq \\ &\leq (\bar{\mu}(A) - \varepsilon) \cdot \eta * \nu^{-1}(\{x \in G : \mu(x^{-1}A) \leq \bar{\mu}(A) - \varepsilon\}) + \bar{\mu}(A) \cdot \eta * \nu^{-1}(\{x \in G : \mu(x^{-1}A) > \bar{\mu}(A) - \varepsilon\}) \leq \\ &\leq (\bar{\mu}(A) - \varepsilon) \cdot (1 - \eta * \nu^{-1}(L^{-1})) + \bar{\mu}(A) \cdot \eta * \nu^{-1}(L^{-1}) \leq \bar{\mu}(A) \end{aligned}$$

implies that  $\eta * \nu^{-1}(L^{-1}) = 1$ . It follows from

$$1 = \eta * \nu^{-1}(L^{-1}) = \int_G \nu^{-1}(y^{-1}L^{-1}) d\eta(y)$$

that for every  $\delta > 0$  there is a point  $y \in G$  such that  $\nu(Ly) = \nu^{-1}(y^{-1}L^{-1}) > 1 - \delta$ . So,  $\text{ls}_{12}(L) = 1$ .

By Lemma 2, the family  $\mathcal{J} = \{B \subset G : \exists E \in [G]^{<\omega} B \subset E(G \setminus L)\}$  is a  $G$ -invariant ideal on  $G$ , which does not contain the set  $L \subset F \cdot \Delta_{\mathcal{I}}(A_i)$  and hence does not contain the set  $\Delta_{\mathcal{I}}(A_i)$ . It follows that  $\text{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) \leq |F| \leq 1/\bar{\mu}(A)$ .  $\square$

## 3. PROOF OF THEOREM 2

Let  $X = A_1 \cup \dots \cup A_n$  be a partition of an ideal  $G$ -space  $(X, \mathcal{I})$ . By Lemma 1, there exists a minimal probability measure  $\mu \in P(X)$  such that  $\mathcal{I} \subset \{A \in \mathcal{B}(G) : \mu(A) = 0\}$ .

For every  $i \in \{1, \dots, n\}$  consider the number  $\bar{\mu}(A_i) = \sup_{x \in G} \mu(xA_i)$  and observe that  $\sum_{i=1}^n \bar{\mu}(A_i) \geq 1$ . There are two cases.

1) For every  $i \in \{1, \dots, n\}$   $\bar{\mu}(A_i) \leq \frac{1}{n}$ . In this case for every  $x \in G$  we get

$$1 = \sum_{i=1}^n \mu(xA_i) \leq \sum_{i=1}^n \bar{\mu}(A_i) \leq n \cdot \frac{1}{n} = 1$$

and hence  $\mu(xA_i) = \frac{1}{n}$  for every  $i \in \{1, \dots, n\}$ . For every  $i \in \{1, \dots, n\}$  fix a maximal subset  $F_i \subset G$  such that  $\mu(xA_i \cap yA_i) = 0$  for any distinct points  $x, y \in F_i$ . The additivity of the measure  $\mu$  implies that

$1 \geq \sum_{x \in F_i} \mu(xA_i) \geq |F_i| \frac{1}{n}$  and hence  $|F_i| \leq n$ . By the maximality of  $F_i$ , for every  $x \in G$  there is a point  $y \in F_i$  such that  $\mu(xA_i \cap yA_i) > 0$  and hence  $xA_i \cap yA_i \notin \mathcal{I}$ . The  $G$ -invariance of the ideal  $\mathcal{I}$  implies that  $y^{-1}x \in \Delta_{\mathcal{I}}(A_i)$  and so  $x \in y \cdot \Delta_{\mathcal{I}}(A_i) \subset F_i \cdot \Delta_{\mathcal{I}}(A_i)$ . Finally, we get  $G = F_i \cdot \Delta_{\mathcal{I}}(A_i)$  and  $\text{cov}(\Delta_{\mathcal{I}}(A_i)) \leq |F_i| \leq n$ .

2) For some  $i$  we get  $\bar{\mu}(A_i) > \frac{1}{n}$ . In this case Theorem 4 guarantees that  $\text{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) \leq 1/\bar{\mu}(A_i) < n$  for some  $G$ -invariant ideal  $\mathcal{J} \not\supset \Delta_{\mathcal{I}}(A_i)$  on  $G$ .

#### 4. APPLYING IDEMPOTENT QUASI-INVARIANT MEASURES

In this section we develop a technique involving idempotent right quasi-invariant measures, which will be used in the proof of Theorem 3 presented in the next section.

A measure  $\mu \in P(G)$  on a group  $G$  will be called *right quasi-invariant* if for any  $y \in G$  there is a positive constant  $c > 0$  such that  $c \cdot \mu(Ay) \leq \mu(A)$  for any subset  $A \subset G$ .

For an ideal  $G$ -space  $(X, \mathcal{I})$  and a measure  $\mu \in P(X)$  consider the set

$$P_{\mathcal{I}}(G; \mu) = \{\lambda \in P(G) : \forall g \in G \ \lambda * \delta_g * \mu \in P_{\mathcal{I}}(X)\}$$

and observe that it is closed and convex in the compact Hausdorff space  $P(G)$ .

**Lemma 3.** *Let  $(X, \mathcal{I})$  be an ideal  $G$ -space with countable acting group  $G$ . If for some measure  $\mu \in P(X)$  the set  $P_{\mathcal{I}}(G; \mu)$  is not empty, then it contains a right quasi-invariant idempotent measure  $\nu \in P_{\mathcal{I}}(G; \mu)$ .*

*Proof.* Choose any strictly positive function  $c : G \rightarrow (0, 1]$  such that  $\sum_{g \in G} c(g) = 1$  and consider the  $\sigma$ -additive probability measure  $\lambda = \sum_{g \in G} c(g) \delta_{g^{-1}} \in P(G)$ . On the compact Hausdorff space  $P(G)$  consider the right shift  $\Phi : P(G) \rightarrow P(G)$ ,  $\Phi : \nu \mapsto \nu * \lambda$ .

We claim that  $\Phi(P_{\mathcal{I}}(G; \mu)) \subset P_{\mathcal{I}}(G; \mu)$ . Given any measure  $\nu \in P_{\mathcal{I}}(G; \mu)$  we need to check that  $\Phi(\nu) = \nu * \lambda \in P_{\mathcal{I}}(G; \mu)$ , which means that  $\nu * \lambda * \delta_x * \mu \in P_{\mathcal{I}}(X)$  for all  $x \in G$ . It follows from  $\nu \in P_{\mathcal{I}}(G; \mu)$  that  $\nu * \delta_{g^{-1}x} * \mu \in P_{\mathcal{I}}(X)$ . Since the set  $P_{\mathcal{I}}(X)$  is closed and convex in  $P(X)$ , we get

$$\nu * \lambda * \delta_x * \mu = \sum_{g \in G} c(g) \cdot \nu * \delta_{g^{-1}} * \delta_x * \mu = \sum_{g \in G} \nu * \delta_{g^{-1}x} * \mu \in P_{\mathcal{I}}(X).$$

So,  $\Phi(P_{\mathcal{I}}(G; \mu)) \subset P_{\mathcal{I}}(G; \mu)$  and by Schauder Fixed Point Theorem, the continuous map  $\Phi$  on the non-empty compact convex set  $P_{\mathcal{I}}(G; \mu) \subset P(G)$  has a fixed point, which implies that the closed set  $S = \{\nu \in P_{\mathcal{I}}(G; \mu) : \nu * \lambda = \nu\}$  is not empty. It is easy to check that  $S$  is a subsemigroup of the convolution semigroup  $(P(G), *)$ . Being a compact right-topological semigroup,  $S$  contains an idempotent  $\nu \in S \subset P_{\mathcal{I}}(G; \mu)$  according to Ellis Theorem [6, 2.6]. Since  $\nu * \lambda = \nu$ , for every  $A \subset G$  and  $x \in G$  we get

$$\nu(A) = \nu * \lambda(A) = \sum_{g \in G} c(g) \cdot \nu * \delta_{g^{-1}}(A) = \sum_{g \in G} c(g) \cdot \nu(Ag) \geq c(x) \cdot \nu(Ax),$$

which means that  $\nu$  is right quasi-invariant. □

**Remark 2.** Lemma 3 does not hold for uncountable groups, in particular for the free group  $F_{\alpha}$  with uncountable set  $\alpha$  of generators. This group admits no right quasi-invariant measure. Assuming conversely that some measure  $\mu \in P(F_{\alpha})$  is right quasi-invariant, fix a generator  $a \in \alpha$  and consider the set  $A$  of all reduced words  $w \in F_{\alpha}$  that end with  $a^n$  for some  $n \in \mathbb{Z} \setminus \{0\}$ . Observe that  $F_{\alpha} = Aa \cup A$  and hence  $\mu(A) > 0$  or  $\mu(Aa) > 0$ . Since  $\mu$  is right quasi-invariant both cases imply that  $\mu(A) > 0$  and then  $\mu(Ab) > 0$  for any generator  $b \in \alpha \setminus \{a\}$ . But this is impossible since the family  $(Ab)_{b \in \alpha \setminus \{a\}}$  is disjoint and uncountable.

In the following lemma for a measure  $\mu \in P(X)$  we put  $\bar{\mu}(A) = \sup_{x \in G} \mu(xA)$ .

**Lemma 4.** *Let  $(X, \mathcal{I})$  be an ideal  $G$ -space and  $\mu \in P(X)$  be a measure on  $X$  such that the set  $P_{\mathcal{I}}(G; \mu)$  contains an idempotent right quasi-invariant measure  $\lambda$ . For a subset  $A \subset X$  and numbers  $\delta \leq \varepsilon < \sup_{x \in G} \lambda * \mu(xA)$  consider the sets  $M_{\delta} = \{x \in G : \mu(xA) > \delta\}$  and  $L_{\varepsilon} = \{x \in G : \lambda * \mu(xA) > \varepsilon\}$ . Then:*

- (1)  $\lambda(gM_{\delta}^{-1}) > (\varepsilon - \delta)/(\bar{\mu}(A) - \delta)$  for any point  $g \in L_{\varepsilon}$ ;
- (2) the set  $M_{\delta}$  does not belong to the  $G$ -invariant Boolean ideal  $\mathcal{J}_{\delta} \subset \mathcal{P}(G)$  generated by  $G \setminus L_{\delta}$ ;
- (3)  $\text{cov}_{\mathcal{J}_{\delta}}(\Delta_{\mathcal{I}}(A)) < 1/\delta$ .

*Proof.* Consider the measure  $\nu = \lambda * \mu$  and put  $\bar{\nu}(A) = \sup_{x \in G} \nu(xA)$  for a subset  $A \subset X$ .

1. Fix a point  $g \in L_{\varepsilon}$  and observe that

$$\begin{aligned} \varepsilon &< \lambda * \mu(gA) = \int_G \mu(x^{-1}gA) d\lambda(x) \leq \\ &\leq \delta \cdot \lambda(\{x \in G : \mu(x^{-1}gA) \leq \delta\}) + \bar{\mu}(A) \cdot \lambda(\{x \in G : \mu(x^{-1}gA) > \delta\}) = \\ &= \delta \cdot (1 - \lambda(gM_{\delta}^{-1})) + \bar{\mu}(A) \lambda(gM_{\delta}^{-1}) = \delta + (\bar{\mu}(A) - \delta) \lambda(gM_{\delta}^{-1}) \end{aligned}$$

which implies  $\lambda(gM_\delta^{-1}) > \gamma := \frac{\varepsilon - \delta}{\bar{\mu}(A) - \delta}$ .

2. To derive a contradiction, assume that the set  $M_\delta$  belongs to the  $G$ -invariant ideal generated by  $G \setminus L_\delta$  and hence  $M_\delta \subset E(G \setminus L_\delta)$  for some finite subset  $E \subset G$ . Then

$$M_\delta \subset E(G \setminus L_\delta) = G \setminus \bigcap_{e \in E} eL_\delta.$$

Choose an increasing number sequence  $(\varepsilon_k)_{k=0}^\infty$  such that  $\delta < \varepsilon \leq \varepsilon_0$  and  $\lim_{k \rightarrow \infty} \varepsilon_k = \bar{\nu}(A)$ . For every  $k \in \omega$  fix a point  $g_k \in L_{\varepsilon_k}$ . The preceding item applied to the measure  $\nu$  and set  $L_\delta$  (instead of  $\mu$  and  $M_\delta$ ) yields the lower bound

$$\lambda(g_k L_\delta^{-1}) > \frac{\varepsilon_k - \delta}{\bar{\nu}(A) - \delta}$$

for every  $k \in \omega$ . Then  $\lim_{k \rightarrow \infty} \lambda(g_k L_\delta^{-1}) = 1$  and hence  $\lim_{k \rightarrow \infty} \lambda(z_k L_\delta^{-1} g) = 1$  for every  $g \in G$  by the right quasi-invariance of the measure  $\lambda$ . Choose  $k$  so large that  $\lambda(z_k L_\delta^{-1} g^{-1}) > 1 - \frac{1}{|E|}\gamma$  for all  $g \in E$ . Then the set  $\bigcap_{g \in E} z_k L_\delta^{-1} g^{-1}$  has measure  $> 1 - \gamma$  and hence it intersects the set  $z_k M_a^{-1}$  which has measure  $\lambda(z_k M_a) \geq \gamma$ . Consequently, the set  $M_a^{-1}$  intersects  $\bigcap_{g \in E} L_\delta^{-1} g^{-1}$ , and the set  $M_a$  intersects  $\bigcap_{g \in E} gL = G \setminus (E(G \setminus L_\delta))$ , which contradicts the choice of the set  $E$ .

3. To show that  $\text{cov}_{\mathcal{J}_\delta}(\Delta_{\mathcal{I}}(A)) \leq 1/\delta$ , fix a maximal subset  $F \subset L_\delta$  such that  $\nu(xA \cap yA) = 0$  for any distinct points  $x, y \in L_\delta$ . The additivity of the measure  $\nu$  guarantees that  $1 \geq \sum_{x \in F} \nu(xA) > |F| \cdot \delta$  and hence  $|F| < 1/\delta$ . On the other hand, the maximality of  $F$  guarantees that for every  $x \in F$  there is  $y \in L_\delta$  such that  $\nu(xA \cap yA) > 0$  and hence  $xA \cap yA \notin \mathcal{I}$  and  $y^{-1}x \in \Delta_{\mathcal{I}}(A)$ . Then  $x \in y \cdot \Delta_{\mathcal{I}}(A) \subset F \cdot \Delta_{\mathcal{I}}(A)$  and hence  $L_\delta \subset F \cdot \Delta_{\mathcal{I}}(A)$ . The inclusion  $G \setminus (F \cdot \Delta_{\mathcal{I}}(A)) \subset G \setminus L_\delta \in \mathcal{J}_\delta$  implies  $\text{cov}_{\mathcal{J}_\delta}(F \cdot \Delta_{\mathcal{I}}(A)) \leq |F| < 1/\delta$ .  $\square$

**Corollary 3.** *Let  $(X, \mathcal{I})$  be an ideal  $G$ -space with countable acting group  $G$  and  $\mu \in P(X)$  be a measure on  $X$  such that the set  $P_{\mathcal{I}}(G; \mu)$  is not empty. For any partition  $X = A_1 \cup \dots \cup A_n$  of  $X$  either:*

- (1)  $\text{cov}(\Delta_{\mathcal{I}}(A_i)) \leq n$  for all cells  $A_i$  or else
- (2)  $\text{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) < n$  for some cell  $A_i$  and some  $G$ -invariant Boolean ideal  $\mathcal{J} \subset \mathcal{P}(G)$  such that  $\{x \in G : \mu(xA) > \frac{1}{n}\} \notin \mathcal{J}$ .

*Proof.* By Lemma 3, the set  $P_{\mathcal{I}}(G; \mu)$  contains an idempotent right quasi-invariant measure  $\lambda$ . Then for the measure  $\nu = \lambda * \mu \in P_{\mathcal{I}}(X)$  two cases are possible:

- 1) Every cell  $A_i$  of the partition has  $\bar{\nu}(A_i) = \sup_{x \in G} \nu(xA_i) \leq \frac{1}{n}$ . In this case we can proceed as in the proof of Theorem 2 and prove that  $\text{cov}(\Delta_{\mathcal{I}}(A_i)) \leq n$  for all cells  $A_i$  of the partition.
- 2) Some cell  $A_i$  of the partition has  $\bar{\nu}(A_i) > \frac{1}{n}$ . In this case Lemma 4 guarantees that  $\text{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) < n$  for the  $G$ -invariant Boolean ideal  $\mathcal{J} \subset \mathcal{P}(G)$  generated by the set  $\{x \in G : \nu(xA_i) \leq \frac{1}{n}\}$ , and the set  $M = \{x \in G : \mu(xA_i) > \frac{1}{n}\}$  does not belong to the ideal  $\mathcal{J}$ .  $\square$

Next, we extend Corollary 3 to  $G$ -spaces with arbitrary (not necessarily countable) acting group  $G$ . Given a  $G$ -space  $X$  denote by  $\mathcal{H}$  the family of all countable subgroups of the acting group  $G$ . A subfamily  $\mathcal{F} \subset \mathcal{H}$  will be called

- *closed* if for each increasing sequence of countable subgroups  $\{H_n\}_{n \in \omega} \subset \mathcal{F}$  the union  $\bigcup_{n \in \omega} H_n$  belongs to  $\mathcal{F}$ ;
- *dominating* if each countable subgroup  $H \in \mathcal{H}$  is contained in some subgroup  $H' \in \mathcal{F}$ ;
- *stationary* if  $\mathcal{F} \cap \mathcal{C} \neq \emptyset$  for every closed dominating subset  $\mathcal{C} \subset \mathcal{H}$ .

It is known (see [5, 4.3]) that the intersection  $\bigcap_{n \in \omega} \mathcal{C}_n$  of any countable family of closed dominating sets  $\mathcal{C}_n \subset \mathcal{H}$ ,  $n \in \omega$ , is closed and dominating in  $\mathcal{H}$ .

For a measure  $\mu \in P(X)$  and a subgroup  $H \in \mathcal{H}$  let

$$P_{\mathcal{I}}(H; \mu) = \{\lambda \in P(H) : \forall x \in H \quad \lambda * \delta_x * \mu \in P_{\mathcal{I}}(X)\}.$$

**Theorem 5.** *Let  $(X, \mathcal{I})$  be an ideal  $G$ -space and  $\mu \in P(X)$  be a measure on  $X$  such that the set  $\mathcal{H}_{\mathcal{I}} = \{H \in \mathcal{H} : P_{\mathcal{I}}(H; \mu) \neq \emptyset\}$  is stationary in  $\mathcal{H}$ . For any partition  $X = A_1 \cup \dots \cup A_n$  of  $X$  either:*

- (1)  $\text{cov}(\Delta_{\mathcal{I}}(A_i)) \leq n$  for all cells  $A_i$  or else
- (2)  $\text{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) < n$  for some cell  $A_i$  and some  $G$ -invariant Boolean ideal  $\mathcal{J} \subset \mathcal{P}(G)$  such that  $\{x \in G : \mu(xA_i) > \frac{1}{n}\} \notin \mathcal{J}$ .

*Proof.* Let  $\mathcal{H}_\forall = \{H \in \mathcal{H}_{\mathcal{I}} : \forall i \leq n \quad \text{cov}(H \cap \Delta_{\mathcal{I}}(A_i)) \leq n\}$  and  $\mathcal{H}_\exists = \mathcal{H}_{\mathcal{I}} \setminus \mathcal{H}_\forall$ . It follows that for every  $H \in \mathcal{H}_\forall$  and  $i \in \{1, \dots, n\}$  we can find a subset  $f_i(H) \subset H$  of cardinality  $|f_i(H)| \leq n$  such that  $H \subset f_i(H) \cdot \Delta_{\mathcal{I}}(A_i)$ . The assignment  $f_i : H \mapsto f_i(H)$  determines a function  $f_i : \mathcal{H}_\forall \rightarrow [G]^{<\omega}$  to the family of all finite subsets of  $G$ . The function  $f_i$  is regressive in the sense that  $f_i(H) \subset H$  for every subgroup  $H \in \mathcal{H}_\forall$ .

By Corollary 3, for every subgroup  $H \in \mathcal{H}_\exists$ , there are an index  $i_H \in \{1, \dots, n\}$  and a finite subset  $f(H) \subset H$  of cardinality  $|f(H)| < n$  such that the set  $J_H = H \setminus (f(H) \cdot (H \cap \Delta_{\mathcal{I}}(A_{i_H})))$  generates the  $H$ -invariant ideal  $\mathcal{J}_H \subset \mathcal{P}(H)$  which does not contain the set  $M_H = \{x \in H : \mu(xA_{i_H}) > \frac{1}{n}\}$ .

Since  $\mathcal{H}_{\mathcal{I}} = \mathcal{H}_\forall \cup \mathcal{H}_\exists$  is stationary in  $\mathcal{H}$ , one of the sets  $\mathcal{H}_\forall$  or  $\mathcal{H}_\exists$  is stationary in  $\mathcal{H}$ .

If the set  $\mathcal{H}_\forall$  is stationary in  $\mathcal{H}$ , then by Jech's generalization [4], [5, 4.4] of Fodor's Lemma, the stationary set  $\mathcal{H}_\forall$  contains another stationary subset  $\mathcal{S} \subset \mathcal{H}_\forall$  such that for every  $i \in \{1, \dots, n\}$  the restriction  $f_i|_{\mathcal{S}}$  is a constant function and hence  $f_i(\mathcal{S}) = \{F_i\}$  for some finite set  $F_i \subset G$  of cardinality  $|F_i| \leq n$ . We claim that  $G = F_i \cdot \Delta_{\mathcal{I}}(A_i)$ . Indeed, given any element  $g \in G$ , by the stationarity of  $\mathcal{S}$  there is a subgroup  $H \in \mathcal{S}$  such that  $g \in H$ . Then  $g \in H \subset f_i(H) \cdot \Delta_{\mathcal{I}}(A_i) = F_i \cdot \Delta_{\mathcal{I}}(A_i)$  and hence  $\text{cov}(\Delta_{\mathcal{I}}(A_i)) \leq |F_i| \leq n$  for all  $i$ .

Now assume that the family  $\mathcal{H}_\exists$  is stationary in  $\mathcal{H}$ . In this case for some  $i \in \{1, \dots, n\}$  the set  $\mathcal{H}_i = \{H \in \mathcal{H}_\exists : i_H = i\}$  is stationary in  $\mathcal{H}_\exists$ . Since the function  $f : \mathcal{H}_\exists \rightarrow [G]^{<\omega}$  is regressive, by Jech's generalization [4], [5, 4.4] of Fodor's Lemma, the stationary set  $\mathcal{H}_i$  contains another stationary subset  $\mathcal{S} \subset \mathcal{H}_i$  such that the restriction  $f|_{\mathcal{S}}$  is a constant function and hence  $f(\mathcal{S}) = \{F\}$  for some finite set  $F \subset G$  of cardinality  $|F| < n$ . We claim that the set  $J = G \setminus (F \cdot \Delta_{\mathcal{I}}(A_i))$  generates a  $G$ -invariant ideal  $\mathcal{J}$ , which does not contain the set  $M = \{x \in G : \mu(xA_i) > \frac{1}{n}\}$ . Assume conversely that  $M \in \mathcal{J}$  and hence  $M \subset EJ$  for some finite subset  $E \subset G$ . By the stationarity of the set  $\mathcal{S}$ , there is a subgroup  $H \in \mathcal{S}$  such that  $E \subset H$ . It follows  $H \cap J = H \setminus (F \cdot (H \cap \Delta_{\mathcal{I}}(A_i))) = H \setminus (f(H) \cdot (H \cap \Delta_{\mathcal{I}}(A_{i_H}))) = J_H$  and

$$M_H = \{x \in H : \mu(xA_i) > \frac{1}{n}\} = H \cap M \subset H \cap EJ = EJ_H \in \mathcal{J}_H,$$

which contradicts the choice of the ideal  $\mathcal{J}_H$ .  $\square$

## 5. PROOF OF THEOREM 3

Theorem 3 is a simple corollary of Theorem 5. Indeed, assume that  $G = A_1 \cup \dots \cup A_n$  is a partition of a group and  $\mathcal{I} \subset \mathcal{P}(G)$  is an invariant ideal on  $G$  which does not contain some countable subset and hence does not contain some countable subgroup  $H_0 \subset G$ . Let  $\mathcal{H}$  be the family of all countable subgroups of  $G$  and  $\mu = \delta_1$  be the Dirac measure supported by the unit  $1_G$  of the group  $G$ . We claim that for every subgroup  $H \in \mathcal{H}$  that contains  $H_0$  the set  $P_{\mathcal{I}}(H; \mu)$  is not empty. It follows from  $H_0 \notin \mathcal{I}$  that the family  $\mathcal{I}_H = \{H \cap A : A \in \mathcal{I}\}$  is an invariant Boolean ideal on the group  $H$ . Then the family  $\{H \setminus A : A \in \mathcal{I}\}$  is a filter on  $H$ , which can be enlarged to an ultrafilter  $\mathcal{U}_H$ . The ultrafilter  $\mathcal{U}_H$  determines a 2-valued measure  $\mu_H : \mathcal{P}(H) \rightarrow \{0, 1\}$  such that  $\mu_H^{-1}(1) = \mathcal{U}_H$ . By the right invariance of the ideal  $\mathcal{I}$ , for every  $A \in \mathcal{I}$  and  $x \in H$  we get  $\mu_H * \delta_x * \mu(A) = \mu_H(Ax) = 0$ , which means that  $\mu_H \in P_{\mathcal{I}}(H; \mu)$ . So, the set  $\mathcal{H}_{\mathcal{I}} = \{H \in \mathcal{H} : P_{\mathcal{I}}(H; \mu) \neq \emptyset\} \supset \{H \in \mathcal{H} : H \supset H_0\}$  is stationary in  $\mathcal{H}$ .

Then by Theorem 5 either

- (1)  $\text{cov}(\Delta_{\mathcal{I}}(A_i)) \leq n$  for all cells  $A_i$  or else
- (2)  $\text{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) < n$  for some cell  $A_i$  and some  $G$ -invariant Boolean ideal  $\mathcal{J} \subset \mathcal{P}(G)$  such that  $A_i^{-1} = \{x \in G : \delta_1(xA_i) > \frac{1}{n}\} \notin \mathcal{J}$ .

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